Ising-like critical phenomena in two-dimensional percolation: series expansion evidence

1986 J. Phys. A: Math. Gen. 19 1459

(http://iopscience.iop.org/0305-4470/19/8/026)

J. Phys. A: Math. Gen. 19 (1986) 1459-1468. Printed in Great Britain

Ising-like critical phenomena in two-dimensional percolation: series expansion evidence

Giancarlo Jug

Process Physics Division, AFRC-Food Research Institute, Colney Lane, Norwich NR4 7UA, UK

Received 14 May 1985, in final form 1 October 1985

Abstract. A recent analytic theory of two-dimensional isotropic percolation indicates that the critical behaviour near p_c is determined by the same renormalisation group fixed point describing the behaviour of the associated dilute Ising model. A specific prediction is that the mean number of clusters, $K(p)$, contains a singularity of the form $K(p) \sim |p-p_c|^2 \ln|\ln|p-p_c||$, rather than the currently accepted form $K(p) \sim |p-p_c|^{2-\alpha}$, with $\alpha = -\frac{2}{3}$. Novel series expansion studies for the site and bond percolation problems on the triangular and simple quadratic lattices, respectively, are presented in support of the new finding, which implies the absence of a separate universality class for two-dimensional percolation processes.

1. Introduction

The scaling theory of percolation processes (for a review see Stauffer (1979), Essam (1980) and Deutscher *et al* (1983)) has been developed extensively over the last decade, borrowing heavily from the familiar description of ordinary critical phenomena. A consistent picture has emerged in which the leading asymptotic behaviour of the various geometric properties of percolation is taken to be as follows:

$$K(p) \sim |p-p_c|^{2-\alpha} \qquad P(p) \sim |p-p_c|^{\beta} \qquad S(p) \sim |p-p_c|^{-\gamma}$$
$$C(p, r) \sim r^{2-d-\eta} \exp(-r/\xi) \qquad \xi(p) \sim |p-p_c|^{-\nu}. \tag{1.1}$$

Here, p is the concentration of the percolating species and the subscript 'c' denotes the critical point, or percolation threshold. Also, $K(p)$ is the mean number of clusters per site, $P(p)$ the percolation probability, $S(p)$ the mean cluster size, $C(p, r)$ the pair connectedness function and $\xi(p)$ the pair connectedness length. The set of critical exponents thus defined, $\alpha, \beta, \gamma, \eta$ and ν, has been the object of numerous theoretical studies which have made use of a variety of approximate techniques. Unfortunately, no rigorous exact results are available for percolation processes on ordinary lattices for dimensions $d > 1$, except for the values of p_c for a few two-dimensional (2D) lattices (in particular, $p_c = \frac{1}{2}$ for both the site percolation problem on the triangular lattice (STR) and the bond percolation problem on the simple quadratic lattice (BSQ) (Sykes and Essam 1964)). However, it is believed that the exponents of a percolation process belong to a universality class of their own. As is well known, this universality class can be identified with that of the $q = 1$ limit of the q-state Potts model (Kasteleyn and

0305-4470/86/081459+10$02.50

Fortuin 1969, Fortuin and Kasteleyn 1972, Wu 1982) or, alternatively, with that of the zero temperature limit of the dilute Ising model on the lattice sustaining the percolation process (Elliott *et al* 1960, Essam 1980, Stinchcombe 1983). This suggests that, for the latter model, a crossover in the critical properties takes place as $T \to 0$ and $p \to p_c$, indicating that the point $T=0$ and $p=p_c$ is a special multicritical point (Stauffer 1975, Lubensky 1977, 1979, Stanley *et al* 1976, Stephen and Grest 1977). Both the mapping of percolation on the Potts model and on the dilute Ising model have been widely exploited in the study of the behaviour near p_c.

In the case of 2D percolation processes, the critical exponents are known to a high degree of accuracy. Indeed, the claim has been advanced (den Nijs 1979, Nienhuis *et al* 1980) that the exponents are now known exactly in the form of rational fractions; for instance, $\alpha = -\frac{2}{3}$ and $\nu = \frac{4}{3}$. These conjectured 'exact' values are the result, for instance, of an assumed mapping of the 2D Potts model onto the 2D Coulomb gas model which in turn, under certain other plausible assumptions, is exactly solvable for the critical behaviour (Nienhuis 1984). The values thus obtained for the q-state Potts model have also been confirmed, for $q > 1$, by recent studies based on the assumed conformal invariance for this model (Friedan *et al* 1984). Nonetheless, for percolation all of these theories, whether approximate or conjectured exact, rely on the basic assumption that the asymptotic behaviour near p_c is given by equation (1.1).

In a recent paper (Jug 1984), I have proposed a novel theoretical tool for studying the behaviour at phase transitions in 2D Ising spin models. The new method is based on a Grassmann path integral (GPI) representation (Samuel 1980) for the 2D Ising model on the simple quadratic lattice, which is known to possess an exact solution (Onsager 1944), and on a perturbative treatment of any non-ideal feature. When applied to the 2D bond-dilute Ising model, the GPI approach appears to contain a faithful description of the percolation limit. To date, this includes an accurate perturbative evaluation of p_c and $K(p)$, as well as the correct behaviour of the critical line $T_c(p)$ near p_c. Furthermore, by taking the continuum limit of the GPI lattice theory, an exact renormalisation group (RG) treatment for the singular behaviour of $K(p)$ near p_c can be developed (Jug 1984). In terms of the $n \to 0$ Grassmann fields ψ^a, the continuum GPI action reads

$$S_{\mathrm{eff}} = \int \mathrm{d}^2x \left[\frac{1}{2}\mathrm{i} \sum_{a=1}^{n} \bar{\psi}^a (m + \partial) \psi^a - g\left(\sum_a \bar{\psi}^a \psi^a \right)^2 \right] \tag{1.2}$$

where $m \propto pt - (pt)_c$, $g \propto 1-p$ and where $t = \tanh(\beta J)$ is the usual Ising thermal variable ($\beta = 1/k_B T$). One can see from equation (1.2) that the RG fixed point $g^* = 0$ provides a description of the critical behaviour all along the critical line $pt = (pt)_c$. In particular, if $p(> p_c)$ is kept fixed, one obtains a specific heat anomaly of the form $C(T) \sim \ln|\ln|T - T_c(p)||$; this result has also been obtained by other authors (Wolff and Zittartz 1983), using techniques other than the RG. If, on the other hand, one sets $T = 0$, then equation (1.2) yields the following singular form for the percolation $K(p)$:

$$K(p) \sim |p - p_c|^2 \ln|\ln|p - p_c|| \tag{1.3}$$

which implies that $\alpha = 0$, just as for the associated dilute Ising exponent. In other words, the form of the mass m in equation (1.2), along with the absence of a new symmetry or of a singular temperature dependence when $T \to 0$ in S_{eff}, implies that the 2D percolation threshold does not represent a multicritical point, once the proper scaling variables are identified. These scaling variables are $t - t_c$ (that is, $T - T_c$ for $T_c > 0$ and $\exp(-2\beta J)$ for $T_c = 0$) and $p - p_c$. According to the GPI theory, in terms of these variables the critical behaviour is the same for both the 2D dilute Ising and the percolation critical points, implying the absence of a separate universality class for 2D percolation processes. This unexpected result is a direct consequence of the GPI approach to the critical properties of the 2D dilute Ising model (Jug 1984), the only assumptions involved being those of the established field-theoretic RG method (Brézin *et al* 1976) which is carried out in an exact fashion in the case of equation (1.2). Thus far, all attempts to demonstrate that equation (1.2) and its present RG solution are incorrect have been ill fated and equation (1.3) stands as an exact result, in the sense of the renormalisation group theory. Note that, owing to the mapping of the percolation problem onto the $q=1$ Potts model, equation (1.2) also implies that the accepted critical behaviour of this model in 2D should be re-examined. The fact that no new symmetry arises in the GPI approach for $T=0$ is in contrast with the results of another field-theoretic RG treatment of the dilute Ising model (Stephen and Grest 1977, Wallace and Young 1978), which is however only appropriate for high space dimensions (Fucito and Parisi 1981). The new finding (Jug 1984) is then probably an accident associated with the topology of the two-dimensional space and with the fact that disorder is a marginal perturbation for the Ising model in $d=2$.

The prediction, equation (1.3), of the GPI approach to 2D percolation and its general implications are rather surprising in view of the body of evidence that has been accumulated in support of the accepted scaling theory, summarised by equation (1.1). However, it should be noticed that the GPI theory differs from any other theory of 2D percolation in that it predicts a marginal RG fixed point and non-power-law singularities. Indeed, although to date equation (1.3) is the only available prediction, it is expected that marginal corrections will also be present for the remaining properties of 2D percolation. It is then possible that, by relying on the assumptions of equation (1.1), all other theories have been deceived by these marginal corrections.

In order to investigate this possibility, I will present in this paper a re-analysis of the series expansions for the mean number of clusters per site, $K(p)$, of 2D percolation, having in view the GPI prediction for the singularity, equation (1.3). This study seems all the more appropriate as, historically, the first successful scaling theory of percolation has arisen from series expansion studies (Essam 1980). In a subsequent paper, the nature of the singularity in $K(p)$ will be investigated through use of a novel numerical method, thus providing a separate independent test of the competing theories.

The remainder of this paper is organised as follows. In § 2 the original series analysis for $K(p)$ by Domb and Pearce (1976) is re-examined and it is shown that the apparent convergence towards $\alpha \sim -\frac{2}{3}$ may be due to the presence of a very weak logarithmic correction, such as in equation (1.3). In § 3 the specific analysis for logarithmic corrections in $K(p)$ is developed for the BSQ percolation problem, and in § 4 the method is extended to the STR case. These two percolation problems lend themselves naturally to this type of analysis, which requires the knowledge of the exact value of p_c (Adler and Privman 1981). Section 5 contains a discussion of the results obtained and my conclusions. A preliminary brief account of this work has already appeared elsewhere (Jug 1985).

2. Search for simple power-law behaviour and its critique

2.1. *The original analysis of Domb and Pearce*

In their original paper, Domb and Pearce (1976, hereafter referred to as DP) assumed for $K(p)$ the conventional power-law singular behaviour, equation (1.1). The series analysis is complicated by the presence of an unphysical singularity in the left half of the complex p plane superimposed to the weak singularity at p_c. As shown by DP, the interference from the unphysical singularity can be minimised by performing the change of variable $u = p(1-p)$, and by analysing the resulting series for $K(p(u))$. Moreover, duality can be built into the method by considering the expansion for

$$B(u) = K(p(u)) - \tfrac{1}{2}\phi(p(u)) = \sum_n B_n u^n$$

where $\phi(p) = p - 3p^2 + 2p^3$ is the matching polynomial (Sykes and Essam 1964) for both BSQ and STR problems. In this way, the dominant singularity is at $u = u_c = p_c(1-p_c) = \frac{1}{4}$. DP employed the ratio test method and considered the convergence of the first-order Neville table extrapolants,

$$G_n = nH_n - (n-1)H_{n-1}$$

where

$$H_n = 1 + n(u_c B_n / B_{n-1} - 1).$$

According to this method, for $n \gg 1$, $G_n = -1 + \alpha/2 + O(n^{-2})$. For the 19-term series relative to the BSQ problem, the original analysis of DP is shown in figure 1 (broken line); it can be seen that this analysis strongly suggests convergence towards the value $\alpha = -\frac{2}{3}$. DP extended their analysis to the 22-term series relative to the STR problem and came to the same conclusion for the value of α.

2.2. Analysis for a model series expansion

Despite the considerable success of the method employed by DP, one could argue, in view of the form advanced by the GPI theory, equation (1.3), that the presence of a very weak logarithmic correction may also lead to a (fictitious) value of $\alpha \neq 0$. In order to see this, I have applied precisely the same method used by DP to a model series constructed by expanding

$$K(p) = (1 - p/p_c)^2 \ln[1 + C \ln(1 - p/p_c)] F(p) = \sum_n K_n p^n \qquad (2.1)$$

for which (technically) $\alpha = 0$, and assuming a simple power-law singularity at $p_c = \frac{1}{2}$. Here, C is an adjustable constant, whilst $F(p)$ is a factor mimicking the unphysical complex singularity present in the original series for $K(p)$, as discussed above. This singularity can be analysed through the change of variable $w = -p(1+p)$, which then reveals a dominant singularity at $w_0 = -p_0(1+p_0) = 0.259$ for the series relative to the BSQ problem. This corresponds to a pair of complex conjugate singularities p_0 and p_0^* and the ratio test reveals good convergence towards $y = 0.3$ for an (assumed) singular behaviour $K(p) \sim |p - p_0|^y$. In order to simulate this behaviour, I have therefore chosen the form

$$F(p) = (1 + p/w_0 + p^2/w_0)^y$$

with the values of w_0 and y given above. The ratio test analysis conducted for the model series in equation (2.1) indeed reveals an initial convergence towards a fictitious value of $\alpha \sim -0.4$. This value appears to be virtually independent of the value of C, which however determines the amplitude of the fluctuations in the G_n. In figure 1 (full curve), the analysis for the case $C = -1.25$ is shown, and one can see that the

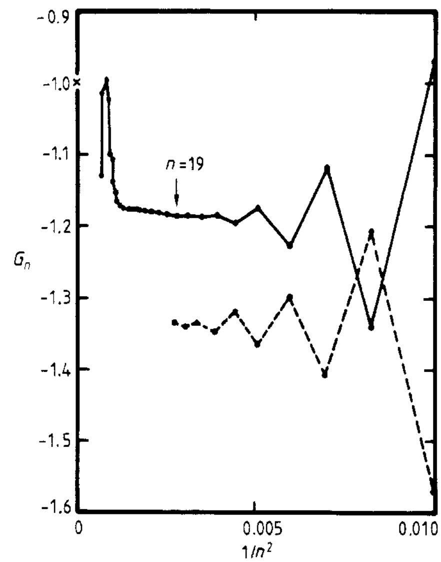

Figure 1. Ratio test first-order Neville table extrapolant analysis for the series expansion of $K(p)$ with an assumed pure power-law singularity. As $n\to\infty$, $G_n=-1+\alpha/2+O(1/n^2)$. Broken curve: original analysis of Domb and Pearce for the mean number of clusters per site of the BSQ percolation problem. Full curve: analysis for the model series expansion of equation (2.1), for which technically $\alpha=0$.

convergence towards the fictitious value of α is very similar to that of the original analysis of DP. Clearly, an identical convergence pattern cannot be easily reproduced, since the true nature of the singularity at p_c is unknown and, in addition, equation (2.1) only models the leading singular behaviour at p_c and p_0. Nevertheless, it appears quite evident, from the simple example given here, that misleading conclusions on the value of a critical exponent can be drawn by assuming a pure power-law behaviour in place of a superimposed weak logarithmic correction.

3. Search for logarithmic corrections: BSQ problem

In view of the above findings, it would seem appropriate to use an alternative method for analysing the series for $K(p)$. The ideal candidate is the method of Adler and Privman (1981), which is specifically designed for the analysis of logarithmic corrections. One assumes that near the singularity of interest $K(p)$ is of the form ($p<p_c$)

$$K(p)=c(p)(p_c-p)^h[\ln(p_c-p)]^{zh}+b(p) \qquad (3.1)$$

where $h=2-\alpha$ and where $c(p)$ contains all other corrections to the leading singular behaviour. The term $b(p)$ in equation (3.1) represents the analytic background in $K(p)$, so that the series

$$K_s(p)=K(p)-b(p)$$

contains only singular terms. Then, by constructing Padé approximants to the series

$$g(p)=h^{-1}(p-p_c)\ln(p_c-p)[K_s'(p)/K_s(p)+h(p_c-p)^{-1}] \qquad (3.2)$$

one can look for the behaviour of $z=z(h)$ as a function of the input value of h. In fact, inserting the form equation (3.1) into equation (3.2), one obtains

$$g(p)=h^{-1}(p-p_c)\ln(p_c-p)c'(p)/c(p)+z$$

which, under the plausible assumption that

$$\lim_{p\to p_c}(p-p_c)\ln(p_c-p)c'(p)/c(p)=0$$

yields

$$g(p_c)=z=z(h). \tag{3.3}$$

For the BSQ problem, the 19-term series expansion is as follows (M F Sykes, private communication):

$$K(p)=p-3p^2+2p^3+p^6-p^7+\tfrac{7}{2}p^8-6p^9+14p^{10}-27p^{11}+\tfrac{115}{2}p^{12}-118p^{13}+265p^{14} -619p^{15}+\tfrac{3041}{2}p^{16}-3715p^{17}+8953p^{18}-21\,061p^{19}+\ldots. \tag{3.4}$$

An analytic background contribution $b(p)$ must be subtracted from this series before equations (3.2) and (3.3) can be used. In the absence of further information, $b(p)$ is chosen in the form predicted by DP:

$$b(p)=K_c+A(p_c-p)+C(p_c-p)^2 \tag{3.5}$$

with K_c, A and C as given by DP. Padé approximants for the $g(p)$ thus constructed have been analysed, and in figure 2 the corresponding approximants for the curve $z=z(h)$ are presented. Only the central group of Padé approximants has been considered; specifically, all approximants $[N, M]$ with $|N-M|=0, 1, 2$ and $14\leqslant N+M\leqslant 18$ are shown in figure 2. According to the general trend, one can conclude that the expected result, $z=0$ for $h=\frac{8}{3}$, is not attained whereas the GPI prediction, $z=0$ for $h=2$, is well supported.

The above analysis can be extended to an assumed form for the singularity in $K(p)$ containing a weaker logarithmic correction of the type

$$K(p)=c(p)(p_c-p)^h\{\ln[-\ln(p_c-p)]\}^{\hat{z}h}+b(p).$$

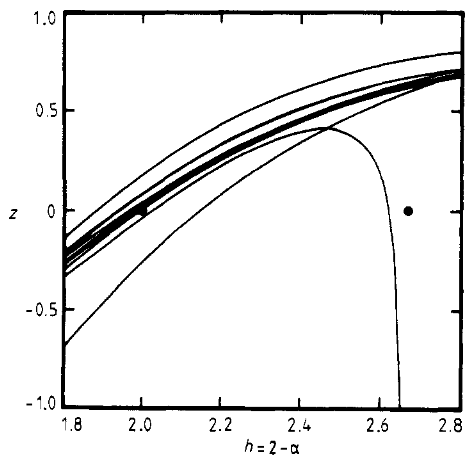

Figure 2. Central group Padé approximants in the search for logarithmic corrections in $K(p)$ for the BSQ percolation problem. The dots denote predictions from the two competing theories being tested.

This singular form can be analysed through the following generalisation of equation (3.2):

$$\hat{g}(p) = h^{-1}(p - p_c)\ln(p_c - p)\ln[-\ln(p_c - p)][K_s'(p)/K_s(p) + h(p_c - p)^{-1}] \tag{3.6}$$

which yields

$$\hat{g}(p_c) = \hat{z} = \hat{z}(h). \tag{3.7}$$

The analysis for equations (3.6) and (3.7) is presented in figure 3, where the central Padé approximants $[N, M]$ to $\hat{z} = \hat{z}(h)$ with $14 \leq N + M \leq 18$ are shown. Again, the general trend supports the GPI prediction, $\hat{z} = \frac{1}{2}$ for $h = 2$, and rules out the conventional result, $\hat{z} = 0$ for $h = \frac{8}{3}$.

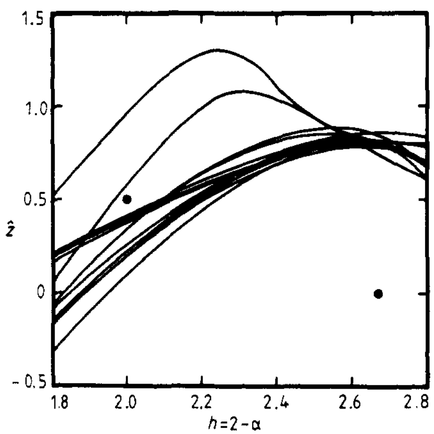

Figure 3. Central group Padé approximants in the search for double logarithmic corrections in $K(p)$ for the BSQ percolation problem. The dots denote predictions from the two competing theories being tested.

4. Search for logarithmic corrections: STR problem

The 22-term low-density expansion for the STR percolation problem is as follows (Sykes *et al* 1976, and private communication):

$$\begin{aligned} K(p) = {} & p - 3p^2 + 2p^3 + p^6 - p^7 + 3p^8 - 4p^9 + 9p^{10} - 15p^{11} + 30p^{12} - 56p^{13} \\ & + 120p^{14} - 248p^{15} + 542p^{16} - 1194p^{17} + 2744p^{18} \\ & - 6267p^{19} + 14\,289p^{20} - 32\,007p^{21} + 71\,529p^{22} + \ldots. \end{aligned} \tag{4.1}$$

It should be stressed that no GPI theory has been developed as yet for this percolation problem. However, the hypothesis of universality implies that at p_c the singularity of the series in equation (4.1) ought to be the same as that for the series in equation (3.4), thus either (1.1) or (1.3). Therefore, the analysis developed in § 3 can be extended to the new series.

The analysis corresponding to equations (3.2) and (3.3) (log corrections) is presented in figure 4, where the central Padé approximants $[N, M]$ to $z = z(h)$ with $14 \leq N + M \leq 21$ are shown. As in the case of the BSQ problem, the general trend supports the GPI prediction ($z = 0$ for $h = 2$) and disproves the conventional theory ($z = 0$ for $h = \frac{8}{3}$).

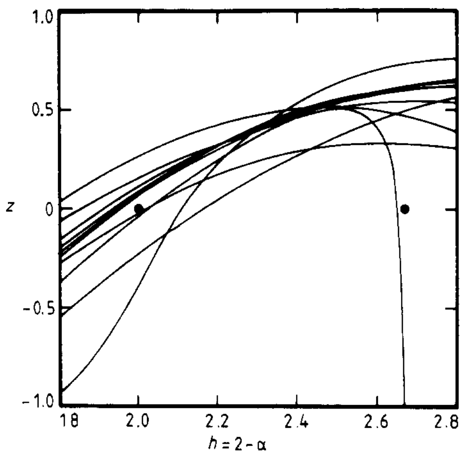

Figure 4. Same as in figure 2, but for the STR percolation problem.

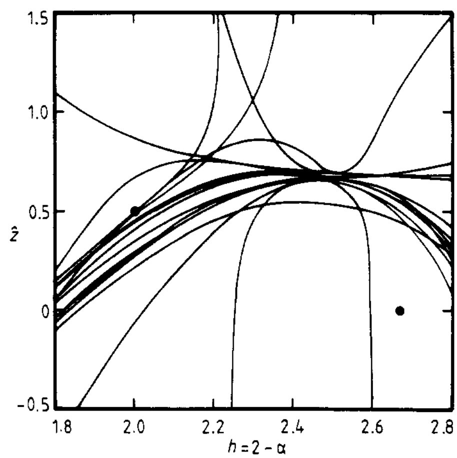

Figure 5. Same as in figure 3, but for the STR percolation problem.

Finally, the analysis corresponding to equations (3.6) and (3.7) (log-log corrections) is summarised in figure 5, which shows the central approximants $[N, M]$ to $\hat{z} = \hat{z}(h)$ with $14 \leqslant N + M \leqslant 21$. Once more the new theory ($\hat{z} = \frac{1}{2}$ for $h = 2$) seems to be better supported than the conventional one ($\hat{z} = 0$ for $h = \frac{8}{3}$).

5. Discussion and conclusions

The results obtained through the series expansion analysis in §§ 3 and 4 seem to support the prediction of the GPI approach for $K(p)$, equation (1.3). Particularly reassuring is the consistency between the outcomes of the analyses for the BSQ and STR problems. At the same time, the inconsistency between the results of the log correction and the log-log correction analyses for $h = \frac{8}{3}$ would rule out both a pure power law and a power law with a superimposed logarithmic correction with a leading exponent $\alpha = -\frac{2}{3}$. Furthermore, the ratio test analysis for the simple model series expansion in § 2.2

indicates that weak marginal corrections to leading power-law behaviour should be taken into account in order to reveal the true critical singularities of 2D percolation. Thus, it is possible that earlier theories, whether approximate or conjectured exact, may have been misled into fictitious exponents by making the assumption of ordinary, simple power-law critical singularities.

It is interesting to remark at this point that the possibility of a breakdown, of the sort discussed in this paper, of the accepted theory of 2D percolation has been detected by other authors in the past. Fucito and Parisi (1981) have indeed found evidence for the breakdown of the $6-\varepsilon$ RG theory of percolation specifically in $d=2$. For dimensions $d=6-\varepsilon$, this theory otherwise predicts the ordinary critical behaviour summarised by equation (1.1). At the same time, Andelman and Berker (1981) have found real-space RG evidence for a marginal operator describing the critical behaviour of the $q=1$ Potts model in $d=2$. The latter finding generated some interest in the possibility of logarithmic corrections to the ordinary power-law exponents of 2D percolation. Such interest was extended to series (Adler and Privman 1981) and numerical (Stauffer 1981) analysis of log corrections for properties such as $S(p)$, $P(p)$ and $\xi(p)$, but the results of this search were inconclusive. Unfortunately, such interest was never extended to $K(p)$, for which a specific theoretical prediction now exists.

In conclusion, the study presented in this paper indicates that the GPI approach to 2D percolation is consistent with series expansions and that its known predictions, including equation (1.3), are correct. It is not clear at present whether these findings would merely imply a breakdown of the hyperscaling relation, $2-\alpha=d\nu$, with $\nu=\frac{4}{3}$ in the accepted theory, or whether the entire current scaling theory of 2D percolation is at fault. Hopefully, further results from the GPI approach will emerge, allowing the construction of a consistent picture of the behaviour near p_c.

Acknowledgments

My thanks are due to M F Sykes for providing me with his unpublished coefficients for the series expansions for $K(p)$, and to H A Duncan, V Privman and an anonymous referee for useful suggestions.

References

Adler J and Privman V 1981 *J. Phys. A: Math. Gen.* **14** L463
Andelman D and Berker A N 1981 *J. Phys. A: Math. Gen.* **14** L91
Brézin E, Le Guillou J and Zinn-Justin J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) pp 125–247
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Deutscher G, Zallen R and Adler J (ed) 1983 *Percolation Structures and Processes. Ann. Israel Phys. Soc.* vol 5 (Bristol: Adam Hilger)
Domb C and Pearce C J 1976 *J. Phys. A: Math. Gen.* **9** L137
Elliott R J, Heap B R, Morgan D J and Rushbrooke G S 1960 *Phys. Rev. Lett.* **5** 366
Essam J W 1980 *Rep. Prog. Phys.* **43** 833
Fortuin C M and Kasteleyn P W 1972 *Physica* **57** 535
Friedan D, Qiu Z and Shenker S H 1984 *Phys. Rev. Lett.* **52** 1575
Fucito F and Parisi G 1981 *J. Phys. A: Math. Gen.* **14** L507
Jug G 1984 *Phys. Rev. Lett.* **53** 9
—— 1985 *Phys. Rev. Lett.* **55** 1343

Kasteleyn P W and Fortuin C M 1969 *J. Phys. Soc. Japan* **26** 11
Lubensky T C 1977 *Phys. Rev.* B **15** 311
—— 1979 *Ill-Condensed Matter. Les Houches Lecture Notes* vol 31, ed R Balian, R Meynard and G Toulouse (Amsterdam: North-Holland) pp 405–75
Nienhuis B 1984 *J. Stat. Phys.* **34** 731
Nienhuis B, Riedel E K and Schick M 1980 *J. Phys. A: Math. Gen.* **13** L189
Onsager L 1944 *Phys. Rev.* **65** 117
Samuel S 1980 *J. Math. Phys.* **21** 2806
Stanley H E, Birgeneau R J, Reynolds P J and Nicoll J F 1976 *J. Phys. C: Solid State Phys.* **9** L553
Stauffer D 1975 *Z. Phys.* **22B** 161
—— 1979 *Phys. Rep.* **54** 1
—— 1981 *Phys. Lett.* **83A** 404
Stephen M J and Grest G S 1977 *Phys. Rev. Lett.* **38** 567
Stinchcombe R B 1983 *Phase Transitions and Critical Phenomena* vol 7, ed C Domb and J L Lebowitz (New York: Academic) pp 151-280
Sykes M F and Essam J W 1964 *J. Math. Phys.* **5** 1117
Sykes M F, Gaunt D S and Glen M 1976 *J. Phys. A: Math. Gen.* **9** 715
Wallace W J and Young A P 1978 *Phys. Rev.* B **17** 2384
Wolff W F and Zittartz J 1983 *Z. Phys.* **52B** 117
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235